Green's Theorem (in R^2)

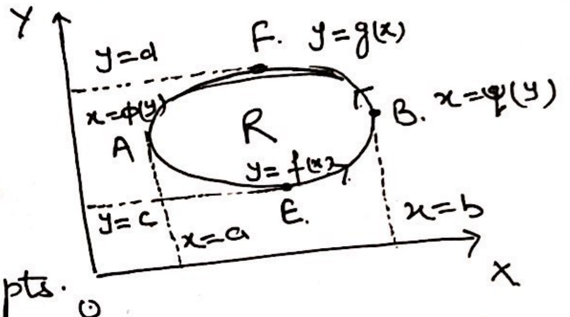
Statement: Let R be a closed region of XY-plane bodded by a simple closed curve C and let M and N be continuous functions of x and y having continuous partial derivative $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R, then

$$\oint (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \text{ , where}$$

C is transversed in positive (anticlockwise) direction.

Proof: We have two cases:

Case(i) The region R be such that any line llal. to either co-ordinate axes meets the boundary C in at most two pts.



This means that R can be represented in both of the forms $R = \{(x,y) : a \le x \le b,\ f(x) \le y \le g(x)\}$

or $R = \{(x,y) : \phi(y) \le x \le \psi(y),\ c \le y \le d\}$

Then $\displaystyle\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left(\int_{y=f(x)}^{g(x)} \frac{\partial M}{\partial y}\,dy\right)dx = \int_a^b \Big[M(x,y)\Big]_{y=f(x)}^{g(x)}\,dx$

$$= \int_a^b [M(x,g(x)) - M(x,f(x))]\,dx$$

$$= -\int_b^a M(x,g(x))\,dx - \int_a^b M(x,f(x))\,dx$$

$$= -\left[\int_{BFA} M(x,y)\,dx + \int_{AEB} M(x,y)\,dx\right]$$

(BFA)

[$\because$ $y=g(x)$ represents curve BFA and $y=f(x)$ represents curve AEB]

$$= -\int_{BFAEB} M(x,y)\,dx = -\oint M\,dx$$

$$\Rightarrow \oint M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy \quad \text{——} (1)$$

Again $\iint\limits_R \frac{\partial N}{\partial x}\,dx\,dy = \int\limits_{y=c}^{d}\left(\int\limits_{x=\phi(y)}^{\psi(y)} \frac{\partial N}{\partial x}\right)dx\,dy.$ (2)

$$= \int_c^d \left[N(x,y)\right]_{x=\phi(y)}^{\psi(y)} dy = \int_c^d [N(\psi(y),y) - N(\phi(y),y)]\,dy$$

$$= \int_c^d N(\psi(y),y)\,dy - \int_c^d N(\phi(y),y)\,dy = \int_c^d N(\psi(y),y)\,dy + \int_d^c N(\phi(y),y)\,dy$$

$$= \int\limits_{EBF} N(x,y)\,dy + \int\limits_{FAE} N(x,y)\,dy$$

[$\because x = \psi(y)$ represents curve EBF and $x = \phi(y)$ represents curve FAE]

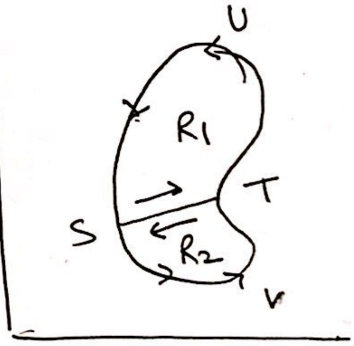
$$= \int\limits_{EBFAE} N(x,y)\,dy = \oint\limits_C N\,dy$$

$$\Rightarrow \oint\limits_C N\,dy = \iint\limits_R \frac{\partial N}{\partial y}\,dx\,dy \quad \text{—— (2)}$$

(1) + (2), we have

$$\oint (M\,dx + N\,dy) = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Case (ii) Region R is bdded by closed curve C s.t. such that lines drawn ‖al. to axes meets the curve C in more than two points.



In this case, by constructing line ST, the region R is divided into two regions R_1 and R_2 s.t. boundary of each meets in at most two points by any line ‖al. to either axes.

Then by case (i), we have

$$\iint\limits_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int\limits_{STUS} (M\,dx + N\,dy) \quad \text{—— (3)}$$

and $$\iint\limits_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int\limits_{SVTS} (M\,dx + N\,dy) \quad \text{—— (4)}$$

Adding (3) and (4), we get

$$\iint_{R_1}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy+\iint_{R_2}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$=\int_{STUS}(M\,dx+N\,dy)+\int_{SVTS}(M\,dx+N\,dy)$$

$$\Rightarrow \iint_{R}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy=\int_{ST}(M\,dx+N\,dy)+\int_{TUS}(M\,dx+N\,dy)$$

$$+\int_{SVT}(M\,dx+N\,dy)+\int_{TS}(M\,dx+N\,dy)$$

$$=\int_{TUS}(M\,dx+N\,dy)+\int_{SVT}(M\,dx+N\,dy)\quad\left[\because \int_{TS}(M\,dx+N\,dy)=-\int_{ST}(M\,dx+N\,dy)\right]$$

$$=\int_{TUSVT}(M\,dx+N\,dy)=\oint_C(M\,dx+N\,dy)$$

$$\Rightarrow \oint_C(M\,dx+N\,dy)=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy.$$

Green thm in vector notation:

In xy plane we have $\overline{r}=x\hat{i}+y\hat{j}$, $\overline{dr}=dx\hat{i}+dy\hat{j}$

Let $\overline{F}=M\hat{i}+N\hat{j}$, then $\overline{F}\cdot\overline{dr}=M\,dx+N\,dy$

and
$$\text{curl }\overline{F}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ M&N&0\end{vmatrix}=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\hat{k}$$

$$\Rightarrow (\text{curl }\overline{F})\cdot\hat{k}=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\hat{k}\cdot\hat{k}=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$$

Hence Green's thm in plane can be written in vector notation as follows:

$$\oint \overline{F}\cdot\overline{dr}=\iint_R(\text{curl }\overline{F})\cdot\hat{k}\,dR\qquad\text{where } dR=dx\,dy.$$
